Validity of Closed Ideals In Algebras of Series of Square Analytic Functions

MUSA SIDDIG and SHAWGY HUSSIEN

1University of Kordofan, Faculty of Science, Department of Mathematics, Sudan.
2Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.

Abstract

We show the validity of a complete description of closed ideals of the algebra,

$$\mathcal{D} \cap \text{lip}_{\alpha_j^2} , 0 < \alpha_j^2 \leq \frac{1}{2}$$

where \( \mathcal{D} \) is the algebra of series of analytic functions satisfying the Lipschitz condition of order \( \alpha_j^2 \) obtained by.\(^{15}\)

Introduction

The Dirichlet space \( \mathcal{D} \) consists of the sequence of square complex-valued analytic functions \( f_j^2 \) on the unit disk \( \mathbb{D} \) with finite Dirichlet integral

$$\sum_j D(f_j^2) := \int_0^1 \sum_j |f_j^2(z)|^2 \, dA(z) < +\infty,$$

where \( dA(z) = \frac{1}{\pi} (1 - e^{-\alpha^2}) \, d(1 - e^{-\alpha^2}) \, d\alpha \),

denotes the normalized area measure on \( \mathbb{D} \).

Equipped with the pointwise algebraic operations and the series of norms

$$\sum_j ||f_j||^2 = \frac{1}{\pi} \int_0^1 \sum_j |f_j^2(x)|^2 \, dx \, dA(z) + p(f_j^2) = \sum_{n=0}^\infty \sum_j (1 + n) |f_j^2(n)|^2,$$

\( \mathcal{D} \) becomes a Hilbert space. For \( 0 < \alpha_j^2 \leq 1 \), let \( \text{lip}_{\alpha_j^2} \) be the algebra of sequence of square analytic functions \( f_j^2 \) on \( \mathbb{D} \) that are continuous on \( \mathbb{D} \) satisfying the Lipschitz condition of order \( \alpha_j^2 \) on \( \mathbb{D} \):

$$\sum_j |f_j(z) - f_j(z - \epsilon)| = \sum_{n=0}^\infty \alpha_j^2 |f_j(z)|^2 \quad (|\epsilon| \to 0).$$

Note that this condition is equivalent to

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\[
\sum_j |(f_j^2)'(z)| = \sum_j \alpha((1 - |z|)^{\alpha_j - 1}) \quad (|z| \to 1^-).
\]

Then, lip $\alpha_j^2$ is a Banach algebra when equipped with series of norms
\[
\Sigma_j \|f_j^2\|_a := \sum_j \|f_j^2\|_a + \sup_j \sum_{j}(1 - |z|)^{\alpha_j} |(f_j^2)'(z)| : z \in D).
\]

Here
\[
\Sigma_j \|f_j^2\|_\infty := \sup_{z \in D} \Sigma_j |f_j^2(z)|.
\]

Unlike as for the case when $0 < \alpha_j^2 \leq 1/4$, the inclusion lip $\alpha_j^2 \subset D$ always holds provided that $1/4 < \alpha_j^2 \leq 1$.

In what follows, let $0 < \alpha_j^2 \leq 1/4$ and define
\[
\mathcal{A}_{\alpha_j^2} := D \cap \text{lip } \alpha_j^2
\]

It is easy to check that $\mathcal{A}_{\alpha_j^2}$ is a commutative Banach algebra when it is endowed with the pointwise algebraic operations and series of norms
\[
\Sigma_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} := \Sigma_j \|f_j^2\|_{a_j} + \Sigma_j D^2(f_j^2), \quad (f_j^2 \in \mathcal{A}_{\alpha_j^2}).
\]

In order to describe the closed ideals in subalgebras of the disc algebra $A(D)$, it is natural to make use of Nevanlinna's factorization theory. For $f_j^2 \in A(D)$ there is a canonical factorization $= C_{f_j^2} U_f^2 O_f^2$,

where $C_{f_j^2}$ is a constant, $U_f^2$ a sequence of square inner functions that is

\[
\Sigma_j |U_j^2| = 1 \ a.e. \ on \ T \ and \ O_f^2
\]

the sequence of square outer functions given by

\[
\sum_j O_f^2(z) = \exp\left(1 \frac{2\pi}{\log|f_j^2|} \sum_j e^{i\theta_j} + \left| \int_0^1 \sum_j e^{i\theta_j} \log|f_j^2(e^{i\theta_j})| \, d\theta_j \right| \right).
\]

Denote by $H^\infty(D)$ the algebra of bounded analytic functions. Note that $\alpha_j^2$ has the so-called F-property if $f_j^2 \in \mathcal{A}_{\alpha_j^2}$ and $U$ is an inner function such that $f_j^2/U \in H^\infty(D)$ then

\[
f_j^2/U \in \mathcal{A}_{\alpha_j^2} \text{ and } \Sigma_j \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}} \leq \Sigma_j c_{\alpha_j^2} \|f_j^2\|_{\mathcal{A}_{\alpha_j^2}}
\]

where $c_{\alpha_j^2}$ is independent of $f_j^2$. Korenblum has described the closed ideals of the algebra $H^\infty(D)$ of sequence of square analytic functions $f_j^2$ such that $f_j^2 \in H^\infty$.

Note that the case of the problem of description of closed ideals appears to be more difficult (see6, 7). Brahim Bouya15 described the structure of the closed ideals of the Banach algebras $\mathcal{A}_{\alpha_j^2}$. More precisely he proved that these ideals are standard in the sense of the Beurling-Rudin characterization of the closed ideals in the disc algebra7, we show the general validation following15

**Theorem**

If $I$ is closed ideal of $\mathcal{A}_{\alpha_j^2}$, then

\[
I = \{f_j^2 \in \mathcal{A}_{\alpha_j^2} : (f_j^2)_{\mathcal{E}_Z} = 0 \ and \ f_j^2/U \in H^\infty(D)\},
\]

where $\mathcal{E}_Z$ is the closed ideal of $\mathcal{A}_{\alpha_j^2}$.
where

$$E_\mathcal{Z} := \{ \zeta \in \mathbb{T} : \sum f_j^2 (\zeta) = 0, \forall f_j^2 \in \mathcal{Z} \}$$

and

$$U_\mathcal{Z} \text{ is greatest common divisor of the inner parts of the non-zero functions in } \mathcal{Z}.$$  

Such characterization of closed ideals can be reduced further to a problem of approximation of outer functions using the Beurling–Carleman–Domar resolvent method. Define $d(\zeta, E)$ to be the distance from $\zeta \in \mathbb{T}$ to the set $E \subset \mathbb{T}$. Suppose that $\mathcal{Z}$ is a closed ideal in $\mathcal{A}_{\mathcal{Z}}$ such that $U_\mathcal{Z} = 1$.

We have $Z_{\mathcal{Z}} = E_{\mathcal{Z}}$, where

$$Z_{\mathcal{Z}} := \{ \zeta \in \mathbb{D} : \sum f_j^2 (\zeta) = 0, \forall f_j^2 \in \mathcal{Z} \}.$$  

Next, for $f_j^2 \in \mathcal{A}_{\mathcal{Z}}$ such that

$$\Sigma_j [f_j^2 (\xi)] \leq M_{\mathcal{Z}} d (\xi, E_\mathcal{Z}) \quad (\xi \in \mathbb{T}),$$

where $M_{\mathcal{Z}}$ is a positive constant depending only on $\mathcal{A}_{\mathcal{Z}}$.

We have $f_j^2 \in \mathcal{Z}$ (see section 3 for more precisions). Now, to show Theorem (1.1) we need Theorem (2.1) below, which states that every function in $\mathcal{A}_{\mathcal{Z}} \setminus \{0\}$ can be approximated in $\mathcal{A}_{\mathcal{Z}}$ by functions with boundary zeros of arbitrary high order.

Theorem

Let $f_j^2$ be a function in $\mathcal{A}_{\mathcal{Z}} \setminus \{0\}$ and let $c \geq 0$.

There exists a sequence of functions

$$\left\{ (g_j^2)_{n=1}^{\infty} \right\} \subset \mathcal{A}(\mathbb{D})$$

such that

For all $n \in \mathbb{N}$, we have $\Sigma_j [f_j^2 (\zeta)] = \Sigma_j [g_j^2 (\zeta)] \in \mathcal{A}_{\mathcal{Z}}$ and

$$\lim_{n \to \infty} \Sigma_j [f_j^2 (\zeta)] = 0$$

where

$$E_{f_j^2} := \{ \xi \in \mathbb{T} : \sum f_j^2 (\xi) = 0 \}$$

To show this Theorem, we give a refinement of the classical Korenblum approximation theory.

Main result on approximation of functions in $\mathcal{A}_{\mathcal{Z}}$

Let $f_j^2 \in \mathcal{A}_{\mathcal{Z}}$ and let $\gamma_n := (a_n, (a + \epsilon) n)_{n \geq 0}$ be the countable collection of the (disjoint open) arcs of $\mathbb{T} \setminus E_{f_j^2}$.

We can suppose that the arc lengths of $\gamma_n$ are less than $1/2$. In what follows, we denote by $\Gamma$ the union of a family of arcs $\gamma_n$. Define

$$\sum (f_j^2) (\xi) = \exp \left\{ \frac{1}{2\pi} \int_{\gamma_n} \frac{1}{1 + \log |f_j^2 (e^{i\theta})|} d\theta \right\}.$$  

The difficult part in the proof of Theorem (1.2) is to establish the following

Theorem

Let $f_j^2 \in \mathcal{A}_{\mathcal{Z}} \setminus \{0\}$ be an outer function such that $\Sigma_j [f_j^2] \leq 1$ and let $\epsilon \geq 1$ and $\epsilon > 0$. Then we have

$$f_j^{2(1+\epsilon)} (\gamma_n) \leq C_{\epsilon} e^{-\epsilon (1+\epsilon)},$$

where $C_{\epsilon} e^{-\epsilon (1+\epsilon)}$ is a positive constant independent of $\Gamma$.

Remark

For a set $S \subset \mathcal{A}(\mathbb{D})$, we denote by $co(S)$ the convex hull of $S$ consisting of the intersection of all convex sets that contain $S$. Set $\Gamma_n = U_{\epsilon \geq 0} \gamma_{n+\epsilon}$ and let $f_j^2$ be as in the Theorem (2.1) It is clear that the sequence

$$(f_j^{2(1+\epsilon)} (\gamma_n)^{2(1+\epsilon)})$$

is contained in $\mathcal{A}_{\mathcal{Z}}$.
converges uniformly on compact subsets of D to 
\[ f_j^{2(1+\varepsilon)} \].

We use (2.1) to deduce, by the Hilbertian structure of D, that there is a sequence \((h^2_j)_{n} \in c_0(f_j^{2(1+\varepsilon)}(f_j^{2(1+\varepsilon)})_\varepsilon=0)\) converging to \( f_j^{2(1+\varepsilon)} \) in D. Also, by [9, section 4], we obtain that

\[(h^2_j)_{n} \text{ converges to } f_j^{2(1+\varepsilon)} \text{ in lip } \alpha_j^2 \text{, for sufficiently large } (1+\varepsilon) \text{ (in fact, we can show that this result remains true for every } \varepsilon \geq 0). \]

Therefore

\[ \sum_j \left\| (h^2_j)_{n} - f_j^{2(1+\varepsilon)} \right\|_{\alpha_j^2} \to 0, \quad n \to \infty \]

Define \( J(F) \) to be the closed ideal of all functions in \( \mathcal{A}_{\alpha_j^2} \) that vanish on. \( F \subseteq D \). In the proof of Theorem (1.2), we need the following classical lemma (see 19), see for instance [9, Lemma 4] and [8, Lemma 24].

**Lemma**

Let \( f_j^{2} \in \mathcal{A}_{\alpha_j^2} \) and \( E' \) be a finite subset of \( T \) such that

\[ \sum_j f_j^{2} |E'| = 0. \quad \text{Let } \varepsilon \geq 0 \]

be given. For every \( \varepsilon > 0 \) there is an outer function \( F \) in \( J(E') \) such that

\[ \sum_j \left\| F f_j^{2} - f_j^{2} \right\|_{\mathcal{A}_{\alpha_j^2}} \leq \varepsilon, \]

\[ |F(\xi)| \leq C \varepsilon^{1+\varepsilon}(\xi, E') \quad (\xi \in T) \]

**Proof of Theorem**

Now, we can deduce the proof of Theorem (1.2) by using Theorem (2.1) and Lemma (2.3). Indeed, let be \( f_j^{2} \) a sequence of functions in \( \mathcal{A}_{\alpha_j^2} \setminus \{0\} \) such that

\[ \sum_j \left\| f_j^{2} \right\|_{\mathcal{A}_{\alpha_j^2}} \leq 1 \]

and let \( \varepsilon > 0 \). For \( \varepsilon \geq 0 \) we have

\[ \sum_j \left( \int f_j^{2} \alpha_j^{1+\varepsilon} - f_j^{2} \right) = \sum_j \left( \int \alpha_j^{1+\varepsilon} - f_j^{2} \right) \]

The F-property of \( \alpha_j^2 \) implies that \( \alpha_j^{2} \in \mathcal{A}_{\alpha_j^2} \).

Then, there exists \( \eta_0 \in \mathbb{N} \) such that

\[ \sum_j \left\| f_j^{2} \alpha_j^{1+\varepsilon} - f_j^{2} \right\|_{\alpha_j^2} < \frac{\varepsilon}{3} \quad (\varepsilon \geq 0) \]

Set \( \Gamma_n = U_{1+\varepsilon} \pi_1 + \alpha_j^2 \leq 1 \) for a given \( \varepsilon \geq 0 \). By Remark (2.2) applied to \( \alpha_j^{2} \) (with \( \varepsilon \geq 0 \)), there is a sequence \( k_{n,1+\varepsilon} \in c_0 \left( \left\{ f_j^{2(1+\varepsilon)} \right\}_{\varepsilon=0}^{\infty} \right) \) such that

\[ \sum_j \left\| \alpha_j^{1+\varepsilon} k_{n,1+\varepsilon} - \alpha_j^{1+\varepsilon} \right\|_{\alpha_j^2} < \frac{1}{1+\varepsilon} \quad (n \in \mathbb{N}, \varepsilon \geq 0) \]

It is clear that

\[ \sum_j \left\| \alpha_j^{1+\varepsilon} f_j^{2} \right\|_{\alpha_j^2} \to 0 \quad (n \to +\infty) \]

Then for every \( \varepsilon > 0 \) we get

\[ \sum_j \left\| \alpha_j^{1+\varepsilon} k_{n,1+\varepsilon} - \alpha_j^{1+\varepsilon} \right\|_{\alpha_j^2} \to 0 \quad (n \to +\infty) \]

So, there is a sequence \( k_{1+\varepsilon} \in c_0 \left( \left\{ f_j^{2(1+\varepsilon)} \right\}_{\varepsilon=0}^{\infty} \right) \) such that

\[ \sum_j \left\| \alpha_j^{1+\varepsilon} k_{1+\varepsilon} - \alpha_j^{1+\varepsilon} \right\|_{\alpha_j^2} \leq \frac{1}{1+\varepsilon} \quad (\varepsilon \geq 0), \]

\[ \sum_j \left\| \alpha_j^{1+\varepsilon} k_{1+\varepsilon} - \alpha_j^{1+\varepsilon} \right\|_{\alpha_j^2} \leq \frac{1}{1+\varepsilon} \quad (\varepsilon \geq 0), \]

We have

\[ \sum_j \left( f_j^{2} \alpha_j^{1+\varepsilon} - f_j^{2} \right) = \sum_j \left( \alpha_j^{1+\varepsilon} - f_j^{2} \right) \left( \alpha_j^{1+\varepsilon} k_{1+\varepsilon} - \alpha_j^{1+\varepsilon} \right) + \sum_j \left( \alpha_j^{1+\varepsilon} k_{1+\varepsilon} - \alpha_j^{1+\varepsilon} \right) \]
Since
\[ \Sigma_j \left\| O_{l_j}^{1/2} k_{l+\epsilon} - f_{l_j}^{1/2} \right\|_{A_{q_j}} \leq \Sigma_j C_{q_j} \left\| f_{l_j}^{1/2} \right\|_{A_{q_j}} \leq \Sigma_j C_{q_j} \]
we obtain
\[ \sum_j \left\| f_j^{1/2} O_{l_j}^{1/2} k_{l+\epsilon} - f_j^{1/2} \right\|_{A_{q_j}} \leq \sum_j C_{q_j} \left( \sum_j \left\| f_j^{1/2} O_{l_j}^{1/2} k_{l+\epsilon} - f_j^{1/2} \right\|_{A_{q_j}} \right) + \sum_j \left\| f_j^{1/2} O_{l_j}^{1/2} k_{l+\epsilon} - f_j^{1/2} \right\|_{A_{q_j}} \]
\[ \leq \sum_j \left\| f_j^{1/2} O_{l_j}^{1/2} k_{l+\epsilon} - f_j^{1/2} \right\|_{A_{q_j}} + \sum_j \left\| f_j^{1/2} - f_j^{1/2} : O_{l_j}^{1/2} k_{l+\epsilon} - O_{l_j}^{1/2} \right\|_{A_{q_j}} \]
\[ \leq \sum_j C_{q_j} \left\| f_j^{1/2} O_{l_j}^{1/2} k_{l+\epsilon} - O_{l_j}^{1/2} \right\|_{A_{q_j}} + \sum_j \left\| f_j^{1/2} - f_j^{1/2} : O_{l_j}^{1/2} k_{l+\epsilon} - O_{l_j}^{1/2} \right\|_{A_{q_j}} \]
\[ \leq \sum_j C_{q_j} \left\| f_j^{1/2} O_{l_j}^{1/2} k_{l+\epsilon} - O_{l_j}^{1/2} \right\|_{A_{q_j}} \]
\[ < \epsilon / 3 \quad (\epsilon \geq 0) \]

Consequently we obtain
\[ \sum_j \left\| f_j^{1/2} O_{l_j}^{1/2} k_{l+\epsilon} F_{1+\epsilon} - f_j^{1/2} \right\|_{A_{q_j}} < \epsilon \quad (\epsilon \geq 0) \]

It is not hard to see that
\[ \sum_j \left\| O_{l_j}^{1/2} k_{l+\epsilon} F_{1+\epsilon} (z) \right\|_{A_{q_j}} \leq \sum_j C_{q_j} d^{1+\epsilon} (\xi, E_{l_j}) \quad (\xi \in \mathbb{T}) \]

Therefore
\[ \sum_j (g_j)_{1+\epsilon} = \sum_j O_{l_j}^{1/2} k_{l+\epsilon} F_{1+\epsilon} \]
is the desired series of sequence, which completes the proof of Theorem (1.2).

**Beurling – Carleman – Domar resolvent method**

Since \( A_{q_j} \subset \text{lip}_{q_j} \), then for all \( f_j \in A_{q_j}, E_{f_j} \)
satisfies the Carleson condition
\[ \int \sum_{1} \log \left( \frac{1}{d(e^{it}, E_{f_j})} \right) dt < +\infty \]

For \( f^2_j \in A_{q_j} \)
we denote by \( B_{f_j} \)
the Blaschke product with zeros \( Z_{f_j} \setminus E_{f_j} \),
where \( Z_{f_j} := \{ z \in \mathbb{D} : \sum_j f_j(z) = 0 \} \).

We begin with following lemma (see\(^{15}\)).

**Lemma**

Let \( \mathcal{Z} \) be a closed ideal of \( A_{q_j} \). Define \( B_{\mathcal{Z}} \)
to be the Blaschke product with zeros \( Z_{\mathcal{Z}} \setminus E_{\mathcal{Z}} \).
There is a sequence of functions \( f^2_j \in \mathcal{Z} \)
such that \( B_{f^2_j} = B_{\mathcal{Z}} \).
Hence it is clear that \((g_j^2)_n\) converges uniformly on compact subsets of \(D\) to 
\[
\Sigma_j f_j^2 = \Sigma_j (g_j^2 / B_{g_j^2}) B_x
\]
and we have \(\Sigma_j B_{f_j^2} = B_x\).

In the sequel we prove that

If we obtain
\[
\sum_j \left\| (g_j^2)_n - f_j^2 \right\|_{\mathcal{A}_{\alpha_j^2}} = 0.
\]

Indeed, by the Cauchy integral formula
\[
(z - 2) = \frac{1}{2\pi i} \int_{\partial D} \frac{g_j(z - 2\epsilon) - g_j(z/|z|)}{4|z|^2} \frac{K_n(z - 2\epsilon)}{d(z - 2\epsilon)} (z \in \mathbb{D})
\]

Then, for \(z = (1 - \epsilon) e^{i\theta} \in \mathbb{D}\)
\[
\sum_j \left\| (g_j^2)_n - f_j^2 \right\|_{\mathcal{A}_{\alpha_j^2}} = 0.
\]

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\]

Then, for \(z = (1 - \epsilon) e^{i\theta} \in \mathbb{D}\)
\[
\sum_j \left\| (g_j^2)_n - f_j^2 \right\|_{\mathcal{A}_{\alpha_j^2}} = 0.
\]
we obtain
\[
\int_{-\pi}^\pi \sum_j \left| g_j^2(e^{i(t+\theta_j)}) - g_j^2(e^{i\theta_j}) \right| (2\pi - 1) \cos t^2 + (1 - \epsilon)^2 dt^2 
\leq \epsilon \int_{|t|<\pi} \sum_j \left| \frac{e^{i\theta_j}}{t^2 + 4(1 - \epsilon)t^2 / \pi^2} \right| dt^2 
+ \frac{\|g_j^2\|_{\alpha_j^2}}{\|l_j\|_{\alpha_j^2}} \int_{|t|<\pi} \sum_j \left| e^{i\theta_j} \right| dt^2 
\leq \frac{\epsilon}{(1 - \epsilon)^2} \frac{1}{\pi} \int_{|t|<\pi} \sum_j \frac{1}{1 + (2u / \pi)^2} du 
+ \frac{\|g_j^2\|_{\alpha_j^2}}{\|l_j\|_{\alpha_j^2}} \int_{|t|<\pi} \sum_j \left| e^{i\theta_j} \right| dt^2 
\leq \epsilon O\left(\frac{1}{\epsilon(1 - \alpha_j^2)}\right) + \frac{\|g_j^2\|_{\alpha_j^2}}{\|l_j\|_{\alpha_j^2}} O\left(\frac{1}{\epsilon(1 - \alpha_j^2)}\right). 
\]

Thus
\[ f_j^2 \in \mathfrak{I} \]

This completes the proof of the lemma.

We can see that
\[ \Sigma_j \left\| (g_j^2)_n \right\|_{\alpha_j^2} O\left(\frac{1}{\epsilon(1 - \alpha_j^2)}\right) \leq \Sigma_j O\left(\frac{1}{\epsilon(1 - \alpha_j^2)}\right). \]

As a consequence of Theorem (1.2), we can show Theorem (1.1) and deduce that each closed ideal of \( \mathcal{A}_{\alpha_j^2} \) is standard. For the sake of completeness, we sketch here the proof, (see [15]).

**Proof of Theorem**

Define \( \gamma \) on \( D \) by \( \gamma(z) = z \) and let \( \pi : \mathcal{A}_{\alpha_j^2} \to \mathcal{A}_{\alpha_j^2} / \mathfrak{I} \) be the canonical quotient map.

Also, let \( f_j^2 / \mathfrak{I} \in \mathcal{H}^\infty(D) \) and \( (f_j^2)_n \)

be the sequence in Theorem (1.2) associated to \( f_j^2 \) with \( \epsilon \geq 2 \). More exactly, we have

\[ \Sigma_j \left( f_j^2 \right)_n = \Sigma_j f_j^2 \left( g_j^2 \right)_n, \]

where

\[ \Sigma_j \left| \left( g_j^2 \right)_n \right|^2 \leq \Sigma_j \left( f_j^2 \right)_n \left| \left( g_j^2 \right)_n \right|^2 \]

Define

\[ \Sigma_j \left( f_j^2 \right)_n = \Sigma_j d^3 \left( \xi, E_{f_j^2} \right) \leq d^3 \left( \xi, E_{f_j^2} \right) \]

Converting to \( f_j^2 \) in \( D \), it is clear that

\[ (h_j^2)_n \in \mathfrak{I} \text{ and } \lim_{n \to +\infty} \Sigma_j \left\| (h_j^2)_n - f_j^2 \right\|_{\alpha_j^2} = 0 \]

Then

\[ \lim_{n \to +\infty} \Sigma_j \left\| (h_j^2)_n - f_j^2 \right\|_{\alpha_j^2} = 0 \]
Then
\[
\sum_j \pi(f_j^2)(\pi(\gamma) - \lambda)^{-1} = \sum_j f_j^2(\lambda)(\pi(\gamma) - \lambda)^{-1} + \sum_j \frac{\pi(L_j(f_j^2))}{n}
\]

It is clear that \((\pi(\gamma) - \lambda)^{-1}\) is an analytic function on \(\mathbb{C}\backslash \mathbb{Z}\).

Note that the multiplicity of the pole \(z_0 \in \mathbb{Z}\cap |\lambda| < 1\) of \((\pi(\gamma) - \lambda)^{-1}\)
is equal to the multiplicity of the zero \(z_0\) of \(U_\lambda\).

Since \(U_\lambda\) divides \(f_j^2\), then according to (3) we can deduce that
\[
\sum_j \pi(f_j^2)(\pi(\gamma) - \lambda)^{-1}
\]
is a series of square analytic functions on \(\mathbb{C}\backslash \mathbb{Z}\). Let \(|\lambda| > 1\), we have
\[
\sum_j \|\pi(f_j^2)(\pi(\gamma) - \lambda)^{-1}\|_{a_{\lambda}^j} \\
\leq \sum_j \|f_j^2\|_{a_{\lambda}^j} \sum_j \|\pi(\lambda)^{-1}\|_{a_{\lambda}^j} |\lambda|^{-n-1} \\
\leq \sum_j \|f_j^2\| \|\pi(\lambda)^{-1}\|_{a_{\lambda}^j} (|\lambda| - 1)^{-n-1}.
\]

By Lemma (3.1), there is \(g_j^2 \in \mathbb{I}\) such that
\[
B_{g_j^2} = B_2. \text{ Let } k = \sum_j f_j^2(g_j^2/B_{g_j^2}).
\]

Then,
\[
k = \sum_j (f_j^2/B_{g_j^2})g_j^2 \in \mathbb{I} \text{ and for } |\lambda| < 1 \text{ we have} \\
k(\lambda)(\pi(\gamma) - \lambda)^{-1} = -\pi(L_\lambda(k)).
\]

Therefore
\[
\sum_j \pi(f_j^2)(\pi(\gamma) - \lambda)^{-1} \leq \sum_j \|f_j^2(\lambda)\|_{a_{\lambda}^j} + \sum_j \frac{C(f_j^2,k)}{\lambda(1-|\lambda|)|\lambda|/B_{g_j^2}(\lambda)} \\
\leq \sum_j C(f_j^2,k)e^{c|\lambda|} (|\lambda| < 1).
\]

We use [14, Lemmas 5.8 and 5.9] to deduce
\[
\sum_j \|\pi(f_j^2)(\pi(\gamma) - \xi)^{-1}\|_{a_{\xi}^j} \leq \sum_j \frac{C(f_j^2,k)}{d(\xi, E^2)} (1 \leq |\xi| \leq 2, \xi \in E^2)
\]

Then, we obtain
\[
\xi \rightarrow \sum_j \|((g_j^2)_{\xi})(\xi)||\pi(f_j^2)(\pi(\gamma) - \xi)^{-1}|| L^\infty(\mathbb{T})
\]

With a simple calculation as in [5, Lemma 2.4], we can deduce that
\[
\sum_j \pi(f_j^2) = \frac{1}{2\pi i} \int \sum_j ((g_j^2)_{\xi})(\xi)||\pi(\lambda)^{-1} - d\xi\]

Denote
\[
\mathbb{I}_{\mathbb{D}}(E_\xi) = \left\{ h_\xi^j \in A(D): (h_\xi^j)_{E_\xi} = 0 \text{ and } h_\xi^j / U_\xi \in A(D) \right\}
\]

From [7, p. 81], we know that \(\mathbb{I}_{\mathbb{D}}(E_\xi)\) has an approximate identity \((e_1 + \epsilon)_{\epsilon > 0} \in \mathbb{I}\) such that
\[
\|e_1 + \epsilon\|_{\infty} \leq 1. \text{ \(\mathbb{I}\) is dense in } \mathbb{I}_{\mathbb{D}}(E_\xi)
\]

with respect to the sup norm \(\|\cdot\|_{\infty}\), so there exists
\[
(u_1 + \epsilon)_{\epsilon > 0} \in \mathbb{I} \text{ with } \|u_1 + \epsilon\|_{\infty} \leq 1 \text{ and}
\]
\[
\lim_{\epsilon \rightarrow \infty} u_1 + \epsilon(\xi) = 1 \text{ for } \xi \in \mathbb{T}\backslash E_\xi. \text{ Therefore}
\]
\[
\sum_j \pi((f_j^2)_{\xi}) = \sum_j \pi((f_j^2)_{\xi} - (f_j^2)_{\xi}u_1 + \epsilon) \rightarrow 0 \text{ as } \epsilon \rightarrow \infty
\]
as \(\epsilon \rightarrow \infty\) Then
\[
(f_j^2)_{\xi} \in \mathbb{I} \text{ and } f_j^2 \in \mathbb{I}.
\]

Note that: if
lim_{n \to \infty} \sum_i |(g^j_i)(n)\xi| = \sum_i |(g^j_i)\xi| \leq 1 \, \text{.}

then,

\sum_j c^{1+\epsilon}(\xi, E_{g^j_i}) = \sum_j d^3(\xi, E_{g^j_i})

Proof of Theorem

The proof of Theorem (2.1) is based on a series of lemmas. In what follows, \( C_{1+\epsilon} \) will denote a positive number that depends only on \( 1+\epsilon \), not necessarily the same at each occurrence. For an open subset \( \Delta \) of \( D \), we put

\[ \sum_j \| (h_j^j)' \|^2_{L^2(\Delta)} = \int \sum_j |(f^j_j)'(z)|^2 \, dA(z) \]

We begin with the following key lemma (see15).

**Lemma (4.1)**

Let \( f^j_j \in \mathcal{A}_{g^j_i} \) be such that

and let \( \epsilon > 0 \) be given. Then

\[ \int \sum_j |f^j_j(e^{it_z^2})|^2 dt_z \leq \sum_j C_{1+\epsilon} \| (f^j_j)' \|^2_{L^2(\gamma)} \]

where \( a, a+\epsilon \in E_{g^j_i} \), \( \gamma = (a, a+\epsilon) \subset T \setminus E_{g^j_i} \),

\[ d(z) := \min(|z-a|, |z-(a+\epsilon)|) \] and \( \Delta_{\gamma} := \{ z \in D : z/|z| \in \gamma \} \)

**Proof**

Let \( e^{it_z^2} \in \gamma \) and define \( z_{t_z^2} := (1-d(e^{it_z^2}))e^{it_z^2} \)

Since \(|\gamma|<1/2\), we obtain \(|z_{t_z^2}| > 1/2\).

We have \( \sum_j \| f^j_j \|^2_{A_{g^j_i}} \leq 1 \)

\[ \sum_j |f^j_j(e^{it_z^2})|^{2(1+\epsilon)} \leq \sum_j 2^{2(1+\epsilon)}( |f^j_j(e^{it_z^2}) - f^j_j(z_{t_z^2})|^{2(1+\epsilon)} + |f^j_j(z_{t_z^2})|^{2(1+\epsilon)} ) \]

By Hölder’s inequality combined with the fact that

\[ \sum_j \| f^j_j \|^2_{A_{g^j_i}} \leq 1 \, , \quad \text{we get} \]

\[ \sum_j |f^j_j(e^{it_z^2}) - f^j_j(z_{t_z^2})|^{2(1+\epsilon)} \leq 2^{2(1+\epsilon)} \int \sum_j |(f^j_j)'(z)|^2 (1-\epsilon)d(1-\epsilon) \]

Hence

\[ \int \sum_j |f^j_j(e^{it_z^2}) - f^j_j(z_{t_z^2})|^{2(1+\epsilon)} \, dt_z \]

\[ \leq 2^{2(1+\epsilon)} \int \sum_j |(f^j_j)'(e^{it_z^2})|^2 (1-\epsilon)d(1-\epsilon) \]

\[ \leq \sum_j 2^{2(1+\epsilon)} \| (f^j_j)' \|^2_{L^2(\gamma)} \] \hspace{1cm} ...(7)

Since \( d(e^{it_z^2}) \leq 1/2 \), we obtain

\[ d(e^{it_z^2}) \leq d(z_{t_z^2}) \leq \sqrt{2}d(e^{it_z^2}) \]

and note that either \( \xi = a \) or \( \xi = a + \epsilon \). Let

\[ z_{t_z^2}(u) = (1-u)z_{t_z^2} + u\xi \quad (0 \leq u \leq 1) \]

With a simple calculation, we can prove that for all \( e^{it_z^2} \in \gamma \)

and for all \( 0 \leq u \leq 1 \), we have

\[ |z_{t_z^2}(u) - w| > \frac{1}{2}(1-u)d(e^{it_z^2}) \]

is the boundary of \( \Delta_{\gamma} \). Then

\[ \Delta_{\gamma} := \{ z \in D : z - z_{t_z^2}(u) \leq \frac{1}{2}(1-u)d(e^{it_z^2})) \in \Delta_{\gamma} \}

for all \( e^{it_z^2} \in \gamma \)

and for all \( 0 \leq u \leq 1 \). Since \( \sum_j |(f^j_j)'(z)| \)

is a series of subharmonic on \( D \), it follows that

\[ \sum_j |(f^j_j)'(z_{t_z^2}(u))| \leq \frac{4}{\pi^2(1-u)^2} \int \sum_j |(f^j_j)'(z)| dA(z) \]

\[ \leq \frac{4}{\pi^2(1-u)^2} \sum_j \| (f^j_j)' \|^2_{L^2(\gamma)} \]
Set $\varepsilon_{(1+\varepsilon)} = 2\alpha_1^2 \varepsilon$.

We have

$$\sum_j f_j^{(1+\varepsilon)}(x_1) = \sum_j f_j^{(1+\varepsilon)}(x_1) - f_j^{(1+\varepsilon)}(\xi) = (1 + \varepsilon)^2 |x_1 - \xi|^2 \int_j f_j(x_1(u))(f_j)(\xi) \, du \leq C_{i, 1 + \varepsilon} \int_j f_j(x_1(u))(f_j)(\xi) \, du$$

Hence

$$\int_j f_j^{(1+\varepsilon)}(x_1) \, dx_1 \leq C \|f_j^{(2+\varepsilon)}\|_{L^2}^2$$

Therefore the result follows from $6, 7, 8$.

In the sequel, we denote by $f_j^{(2+\varepsilon)}$ a series of square outer functions in $A_{\alpha_1^2}$ such that

$$\sum_j \|f_j^{(2+\varepsilon)}\|_{A_{\alpha_1^2}} \leq 1$$

and we fix a constant $1 + \varepsilon, 0 < \varepsilon \leq 1$. By [9, Theorem B], we have

$$f_j^{(2+\varepsilon)}(f_j^{(2+\varepsilon)}) \in \text{lip}_{\alpha_1^2}$$

and

$$\sum_j \|f_j^{(2+\varepsilon)}(f_j^{(2+\varepsilon)})\|_{L^p} \leq C_{1+\varepsilon, 1+\varepsilon}.$$

To prove Theorem (2.1) we need to estimate the integral

$$\int_\Delta \sum_j \|f_j^{(2+\varepsilon)}(f_j^{(2+\varepsilon)})\|^2 \, dA(x).$$

Define

$$\sum_j \sum_{j=1}^n \sum_{j=1}^{n-1} \frac{e^{i\theta_j}}{(a^{i\theta_j} - z)^2} \log|f_j^{(2+\varepsilon)}| \, d\theta_j.$$

Clearly we have

$$\sum_j \sum_{j=1}^n \sum_{j=1}^{n-1} \frac{e^{i\theta_j}}{(a^{i\theta_j} - z)^2} \log|f_j^{(2+\varepsilon)}| \, d\theta_j$$

and

$$\sum_j \sum_{j=1}^n \sum_{j=1}^{n-1} \frac{e^{i\theta_j}}{(a^{i\theta_j} - z)^2} \log|f_j^{(2+\varepsilon)}| \, d\theta_j = C_{1+\varepsilon, 1+\varepsilon}.$$

Hence, by (11) we get

$$\int_\Delta \sum_j \|f_j^{(2+\varepsilon)(f_j^{(2+\varepsilon)})}\|^2 \, dA(x) \leq C_{1+\varepsilon, 1+\varepsilon}.$$
and put $\varepsilon_{1+\varepsilon} = 2\alpha_1^2\varepsilon$. We have

$$
\sum_{j} \left( 1 - \varepsilon \right) \left| f_j' \left( (1 - \varepsilon) e^{it^2} \right) - f_j' \left( e^{it^2} \right) \right|^2 + \sum_{j} \left( 1 - \varepsilon \right) \left| f_j' \left( (1 - \varepsilon) e^{it^2} \right) - f_j' \left( (1 - \varepsilon) e^{it^2} \right) \right|^2 \leq (1 - \varepsilon) \varepsilon_{1+\varepsilon} \sum_{j} \left| f_j' \left( (1 + \varepsilon) e^{it^2} \right) \right|^2 dA(z) \leq (1 - \varepsilon) \varepsilon_{1+\varepsilon} \sum_{j} \left| f_j' \left( (1 + \varepsilon) e^{it^2} \right) \right|^2 dA(z).
$$

Therefore

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) = \int_{\Delta_{\varepsilon}} \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) \leq \int_{\Delta_{\varepsilon}} \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) \leq \int_{\Delta_{\varepsilon}} \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z).
$$

This completes the proof.

Now, we can state the following result (see 15).

**Lemma (4.3)**

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) = \sum_{j} \left[ C_{1+\varepsilon} \left\| f_j' \right\|_{L^2(\Delta_{\varepsilon})} \right]^2.
$$

**Proof**

By Cauchy’s estimate, it follows that

$$
\Sigma_1 \left| f_j' \left( (1 - \varepsilon) e^{it^2} \right) \right|^2 \leq \frac{1}{\varepsilon}
$$

Using Lemma (4.2), we get

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) \leq \int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 dA(z) \leq \int_{\Delta_{\varepsilon}} \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z).
$$

Using Lemma (4.1), we obtain

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) = \frac{1}{\varepsilon} \int_{\Delta_{\varepsilon}} \left| f_j' \left( \frac{(1 - \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z).
$$

The result of our lemma follows by combining the estimates 14 and 15.

The integral on the region $\Delta_{\varepsilon}$. In this subsection, we estimate the integral

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z)
$$

Before this, we make some remarks. For $z \in D$, define

$$
a_{\varepsilon}(z) = \frac{1}{2\pi} \int_{\Delta_{\varepsilon}} \frac{-\log \left| f_j' \left( e^{it^2} \right) \right|^2}{\left| e^{it^2} - z \right|^2} d\theta,
$$

Using the equation 10, it is easy to see that

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) \leq 4 \int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z).
$$

Using the equation 11, it is clear that

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) \leq 2 \int_{\Delta_{\varepsilon}} \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z).
$$

Then

$$
\int_{\Delta_{\varepsilon}} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z) \leq 2 \int_{\Delta_{\varepsilon}} \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z).
$$

Since $\log |f_j'| \in L^1(\Delta_{\varepsilon})$, we have

$$
a_{\varepsilon}(z) \leq \frac{C}{d^2(\Delta_{\varepsilon})}, \quad (z \in \Delta_{\varepsilon}).
$$

Given such inequality, it is not easy to estimate immediately the integral of the series of functions

$$
\sum_{j} \left| f_j' \left( z \right) \right|^2 \left| f_j' \left( \frac{(1 + \varepsilon) e^{it^2}}{x} \right) \right|^2 dA(z)
$$

In what follows, we give a partition of $\Delta_{\varepsilon}$ into three parts so that one can estimate the integral.
\[ \int \sum_j |f_j^2(z)|^{2(1+\epsilon)} a_j^2(z) dA(z) \] on each part. Let \( \Delta_j^3 \),

three situations are possible:

\[ a_j(z) \leq B \frac{\log(d(z)))}{d(z)} \] ...

(20)

\[ 8 \frac{\log(d(z)))}{d(z)} < a_j(z) < 8 \frac{\log(d(z)))}{\epsilon} \] ...

(21)

\[ 8 \frac{\log(d(z)))}{\epsilon} \leq a_j(z) \] ...

(22)

We can now \( \Delta_j^3 \) into the following three parts

\[ \Delta_j^{31} = \{ z \in \Delta_j^3 : z \text{ satisfying } (20) \}, \]

\[ \Delta_j^{32} = \{ z \in \Delta_j^3 : z \text{ satisfying } (21) \}, \]

\[ \Delta_j^{33} = \{ z \in \Delta_j^3 : z \text{ satisfying } (22) \}. \]

The integral on the regions \( \Delta_j^{31} \) and \( \Delta_j^{32} \). In this case we begin by the following (see\(^{19}\))

Lemma (4.4):

\[ \int \sum_j \int \sum_j f_j(z) a_j(z) dA(z) \leq \sum_j \sum_j \| f_j(z) \|_{L^2(\Delta_j)}^2 \]

Proof

Using Lemma (4.2), we get

\[ \int \sum_j \int \sum_j f_j(z) a_j(z) dA(z) \]

\[ \leq 2^{(1+\epsilon)} \int \sum_j \int \sum_j f_j(z) |f_j(z) - f_j(z/|z|)|^{(1+\epsilon)} a_j(z) dA(z) \]

\[ + 2^{(1+\epsilon)} \int \sum_j \int \sum_j f_j(z) |f_j(z/|z|)|^{(1+\epsilon)} a_j(z) dA(z) \]

\[ \leq c_{1+\epsilon} \int \sum_j \int \sum_j f_j(z) |f_j(z) - f_j(z/|z|)|^{(1+\epsilon)} a_j(z) dA(z) \]

\[ + c_{1+\epsilon} \int \sum_j \int \sum_j f_j(z/|z|)^{(1+\epsilon)} a_j(z) dA(z) \]

\[ \leq \sum_j \sum_j \| f_j(z) \|_{L^2(\Delta_j)}^2 + c_{1+\epsilon} \int \sum_j \int \sum_j f_j(z) |f_j(z/|z|)|^{(1+\epsilon)} a_j(z) dA(z) \]

Let \( e^{it} \in \gamma \) and denote by \( (z-2e)_+ \) the point of

\[ \partial \Delta_j^3 \cap \mathbb{D} \]

such that

\[ (z-2e)_+/(z-2e)_+ = e^{it} \] We have

\[ |e^{it} - (z-2e)| = 1 - |(z-2e)| \leq d(e^{it}) \]

Before doing this, we begin with some lemmas (see\(^{19}\)).

The next one is essential for what follows. Note that a similar result is used by different authors: Korenblum,\(^8\)

Matheson,\(^9\) Shamoyan,\(^11\) and Shirokov.\(^13, 12\)

Lemma (4.6)

Let \( z \in \Delta_j^{32} \) and let \( \mu(z) = 1 - \frac{8|\log(d(z)))|}{a_j(z)} \). Then

\[ \sum_j |f_j^2(z) \mu(z)| \leq d^2(z) \]

Using Lemma (4.1), we get

\[ L_{2,1} \leq \sum_j \int \sum_j f_j(z)^2 dA(z) \leq CA(\Delta_j) \]

where \( A(\Delta_j) \) is the area measure of \( \Delta_j \).

Proof

Set

\[ \Lambda_j = \{ \Gamma \setminus \Lambda \text{ for } \gamma \in \Gamma \}. \]

Using,\(^{19}\) we obtain the result.

The integral on the region \( \in \Delta_j^{31} \). Here, we will give an estimate of the following integral

\[ \int \sum_j \int \sum_j f_j(z) a_j(z) dA(z). \]

Before doing this, we begin with some lemmas (see\(^{19}\)).

The next one is essential for what follows. Note that a similar result is used by different authors: Korenblum,\(^8\)

Matheson,\(^9\) Shamoyan,\(^11\) and Shirokov.\(^13, 12\)

Lemma (4.6)

Let \( z \in \Delta_j^{32} \) and let \( \mu(z) = 1 - \frac{8|\log(d(z)))|}{a_j(z)} \). Then

\[ \sum_j |f_j^2(z) \mu(z)| \leq d^2(z) \]
Proof
Let $z \in \Delta_\rho$ and let $\mu < 1$. We have

$$\sum_\{z \in \Delta_\rho\} |f_j^\circ (\mu z)| \leq \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[ \sum_\{e^{i\theta - \mu \theta} \} \log |f_j^\circ (e^{i\theta})| \right] d\theta \right\} \leq \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[ \sum_\{e^{i\theta - \mu \theta} \} \log |f_j^\circ (e^{i\theta})| \right] d\theta \right\} \leq \exp \left\{ (1 - \mu(1 - \epsilon)) \inf_{\theta \in \Delta_\rho} |e^{i\theta} - \mu z|^2 \right\} \leq \exp \left\{ -(1 - \mu(1 - \epsilon)) \inf_{\theta \in \Delta_\rho} |e^{i\theta} - \mu z|^2 \right\},$$

For $z \in \Delta_\rho^2$ it is clear that $1 - \mu z \leq d(z) \leq 1 - \mu^2$, and so $e^{i\theta z} \in \Lambda_\rho$. Then

$$\inf_{\theta \in \Delta_\rho} \left| e^{i\theta z} - \mu z \right|^2 \geq \frac{1}{2} \quad (z \in \Delta_\rho^2).$$

Thus,

$$\sum_\{z \in \Delta_\rho^2\} |f_j^\circ (\mu z)| \leq \exp \left\{ \frac{1}{4} (1 - \mu^2) \right\} \quad (z \in \Delta_\rho^2).$$

Then, we have

$$\sum_\{z \in \Delta_\rho^2\} |f_j^\circ (\mu z)| \leq \exp \left\{ \frac{1}{4} (1 - \mu^2) \frac{d(z)}{\mu} \right\} \quad (z \in \Delta_\rho^2).$$

For $\epsilon > 0$ define $\gamma_{(1 - \epsilon)} = \{ z \in \mathbb{B} : |z| = 1 - \epsilon \}$ and $z/\gamma \in \gamma$. Without loss of generality, we can suppose that $z \in \Delta_\rho^2$. We need the following (see (15)).

Note that: we deduce that

$$\sum_\{z \in \Delta_\rho^2\} |f_j^\circ (\mu z)| \leq \frac{c^2}{\log |z|} \quad (z \in \Delta_\rho^2).$$

Lemma (4.7)
Let $\epsilon > 0$. Then

$$\sum_\{z \in \Delta_\rho^2\} |f_j^\circ (\mu z)| \leq \frac{\epsilon}{\log |z|} \quad (z \in \Delta_\rho^2).$$

Proof
Using (19) and Lemmas (4.6) and (4.7), we find that

$$\sum_\{z \in \Delta_\rho^2\} |f_j^\circ (\mu z)| \leq \frac{c^2}{\log |z|} \quad (z \in \Delta_\rho^2).$$

Where

$$S_{(1 - \epsilon)} = \left\{ z \in \mathbb{B} : 0 \leq |z - \epsilon| \leq (1 - \epsilon) \right\} \quad (z \in \gamma).$$

The proof is therefore completed.

The last result that we need before giving the proof of Theorem (2.1) is the following one (see (15)).

Lemma (4.8)
Let $\epsilon > 0$. Then

$$\sum_\{z \in \Delta_\rho^2\} |f_j^\circ (\mu z)| \leq \frac{\epsilon}{\log |z|} \quad (z \in \Delta_\rho^2).$$

Proof
Using (19) and Lemmas (4.6) and (4.7), we find that

$$\sum_\{z \in \Delta_\rho^2\} |f_j^\circ (\mu z)| \leq \frac{c^2}{\log |z|} \quad (z \in \Delta_\rho^2).$$

Where

$$S_{(1 - \epsilon)} = \left\{ z \in \mathbb{B} : 0 \leq |z - \epsilon| \leq (1 - \epsilon) \right\} \quad (z \in \gamma).$$

The proof is therefore completed.

The last result that we need before giving the proof of Theorem (2.1) is the following one (see (15)).
This completes the proof of the lemma.

Conclusion
Now, according to (18) and Lemmas (4.4), (4.5) and (4.8), we obtain

\[ \int_{\Omega} \left( \sum_{j=1}^{n} |f_j(z)|^2 \right)^{2(1+\varepsilon)} \left| \left( \frac{\partial f_j}{\partial n} \right)'(z) \right|^2 dA(z) \]

\[ \leq \sum_{j=1}^{n} \left( \sum_{k=1}^{m} |f_j(z)|^2 \right)^{2(1+\varepsilon)} \left| \left( \frac{\partial f_j}{\partial n} \right)'(z) \right|^2 dA(z) \]

Combining this with Lemma (4.3), we deduce that

\[ \int_{\Omega} \left( \sum_{j=1}^{n} |f_j(z)|^2 \right)^{2(1+\varepsilon)} \left| \left( \frac{\partial f_j}{\partial n} \right)'(z) \right|^2 dA(z) \leq \sum_{j=1}^{n} C_{1+\varepsilon} \| f_j \|^2_{L^1(\Omega)} + C A(\Lambda). \]

Hence

\[ \int_{\Omega} \left( \sum_{j=1}^{n} |f_j(z)|^2 \right)^{2(1+\varepsilon)} \left| \left( \frac{\partial f_j}{\partial n} \right)'(z) \right|^2 dA(z) \]

\[ = \sum_{j=1}^{n} \left( \sum_{k=1}^{m} |f_j(z)|^2 \right)^{2(1+\varepsilon)} \left| \left( \frac{\partial f_j}{\partial n} \right)'(z) \right|^2 dA(z) \]

\[ \leq \sum_{j=1}^{n} C_{1+\varepsilon} \sum_{k=1}^{m} \left( \sum_{k=1}^{m} |f_j(z)|^2 \right)^{2(1+\varepsilon)} \left| \left( \frac{\partial f_j}{\partial n} \right)'(z) \right|^2 dA(z) \]

This completes the proof of Theorem (2.1)

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Conflict of Interest
The authors do not have any conflict of interest.

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