Task Scheduling Problem Using Fuzzy Graph

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ABSTRACT

The concept of obtaining fuzzy sum of fuzzy colorings problem has a natural application in scheduling theory. The problem of scheduling N jobs on a single machine and obtain the minimum value of the job completion times is equivalent to finding the fuzzy chromatic sum of the fuzzy graph modeled for this problem. The aim of this paper is to solve task scheduling problems using fuzzy graph.

Keywords: Fuzzy Graph, k-fuzzy coloring, Chromatic fuzzy sum of Graph, \( \Gamma \)-chromatic sum of Graph.

INTRODUCTION

The field of mathematics plays vital role in various field. One of the important areas in Mathematics is graph theory, which is used in several models. The origin of graph theory started with Konigsberg bridge problem, in 1735. It was a long standing problem until solved by Leonhard Euler, by means of graph. The colouring problem consists of determining the chromatic number of a graph and an associated colouring function. Let G be a simple graph with n vertices. A colouring of the vertices of G is a mapping f: V (G) \( \not\rightarrow \) N, such that adjacent vertices are assigned different colours. The chromatic sum of a graph introduced in ⁵ is defined as the smallest possible total over all vertices that can occur among all colourings of G. Senthilraj S.¹⁰ generalize these concepts to fuzzy graphs. He define fuzzy graphs with fuzzy vertex set and fuzzy edge set and generalize the concept of the chromatic joins and chromatic sum of a graph to fuzzy graphs and define the fuzzy chromatic sum of fuzzy graph. Author consider the problem of scheduling N jobs on a single machine and obtain the minimum value of the job completion times which is equivalent to finding the fuzzy chromatic sum of the fuzzy graph modeled for this problem by considering the example of scheduling 6 jobs on a single machine.

In this paper we generalize the above said result by considering the case of scheduling 8 tasks on a single machine and obtain a minimum value of the task completion time.

Preliminaries and results

Definition 2.1:¹¹

A fuzzy set \( A \) defined on a non empty set \( X \) is the family \( A = \{(x, \mu_A(x))| x \in X\} \), where \( \mu_A: X \rightarrow I \) is the membership function. In classical fuzzy set theory the set \( I \) is usually defined as the interval [0,1] such that

\[ \mu_A(x) = \begin{cases} 0 : x \notin X \\ 1 : x \in X \end{cases} \]
It takes any intermediate value between 0 and 1 represents the degree in which \( x \in A \). The set \( I \) could be discrete set of the form \( \{0, 1, \ldots, k\} \) where \( \mu_A(x) < \mu_A(x') \) indicates that the degree of membership of \( x \) to \( A \) is lower than the degree of membership of \( x' \).

**Definition 2.2**

A fuzzy graph \( \hat{G} = (V, \sigma, \mu) \) is called a fuzzy graph on \( V \) where \( \sigma \) and \( \mu \) are fuzzy sets on \( V \) and \( E \), respectively, such that \( \mu(uv) \leq \sigma(u) \land \sigma(v) \) for all \( u, v \in V \) and \( uv \in E \). For fuzzy graph \( \hat{G} = (V, \sigma, \mu) \) the elements \( V \) and \( E \) are called set of vertices and set of edges of \( G \) respectively.

**Definition 2.3**

A fuzzy graph \( \hat{G} = (V, \sigma, \mu) \) is called a complete fuzzy graph if \( \mu(uv) = \sigma(u) \land \sigma(v) \) for all \( u, v \in V \) and \( uv \in E \). We denote this complete fuzzy graph by \( \hat{G}_{k} \).

**Definition 2.4**

Two vertices \( u \) and \( v \) in \( \hat{G} \) are called adjacent if \( (\frac{1}{2})[\sigma(u) \land \sigma(v)] \leq \mu(uv) \).

**Definition 2.5**

The edge \( uv \) of \( \hat{G} \) is called strong if \( u \) and \( v \) are adjacent. Otherwise it is called weak.

**Definition 2.6**

A family \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \) of fuzzy sets on \( V \) is called a \( k \)-fuzzy coloring of \( G = (V, \sigma, \mu) \) if

- a) \( \land \Gamma = \sigma \)
- b) \( \gamma_i \land \gamma_j = 0 \)
- c) For every strong edge of \( \hat{G}, \gamma_i(u) \land \gamma_i(v) = 0 \) for \( 1 \leq i \leq k \)

The above definition of \( k \)-fuzzy coloring was defined by the authors Eslahchi and Onagh on fuzzy set of vertices.

**Definition 2.7:**

The least value of \( k \) for which has a fuzzy coloring, denoted by \( \chi^*(G) \), is called the fuzzy chromatic number of \( G \).

**Definition 2.8:**

For a \( k \)-fuzzy coloring \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \) of a fuzzy graph \( G \), chromatic fuzzy sum of \( G \) denoted by is defined as

\[
\Sigma(G) = \sum_{x \in C_i} \theta_{i}(x) + 2 \sum_{x \in C_{i+1}} \theta_{i}(x) + \ldots + k \sum_{x \in C_k} \theta_{i}(x)
\]

Where,

- \( C_i = \text{supp} \gamma_i \) and \( \theta_{i}(x) = \max\{\sigma(x) + \mu(xy) / y \in C_i\} \)

**Definition 2.9**

The chromatic fuzzy sum of \( G \) denoted by is defined as follows \( \Sigma(G) = \min\{\Sigma(G) / \Gamma \) is fuzzy colouring\}.

The number of fuzzy coloring of \( G \) is finite and so there exist a fuzzy \( \Gamma_0 \) which is called minimum fuzzy coloring of \( G \) such that \( \Sigma(G) = \Sigma_{\gamma_0}(G) \).

**Theorem 2.1**

Let \( G \) be a fuzzy graph and \( \Gamma_0 = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \) is minimum fuzzy sum coloring of \( G \). Then

\[
\sum_{x \in C_i} \theta_{i}(x) \geq \sum_{x \in C_{i+1}} \theta_{i}(x) \geq \ldots \geq \sum_{x \in C_k} \theta_{i}(x)
\]

**Theorem 2.2**

For a fuzzy graph \( \hat{G} = (V, \sigma, \mu), \Sigma(G) \leq 3/4[(\chi^*(G) + 1)h(\sigma)|V|] \), where \( h(\sigma) \) is height of \( \sigma \) and \( |V| \) is cardinality of \( v \).

**Remarks:**

2.1 Let \( \hat{G} = (V, \sigma, \mu) \) be a connected fuzzy graph with strong edges. Then the lower bound for \( \Sigma(G) \) is \( w\sqrt{8e} \) is, where \( w = \max\{\sigma(x) + \mu(xy) / 0, x \in V, (x, y) \text{ is weak edge of } G\} \).

2.2 The fuzzy chromatic sum lies between \( w\sqrt{8e} \) and \( 3/4[(\chi^*(G) + 1)h(\sigma)|V|] \).

**RESULTS AND DISCUSSIONS**

**Result 3.1:** Find a minimum value of the task completion time for scheduling 8 tasks on a single machine.

Assume that at any time the machine is capable to perform any number of tasks and these tasks are independent or conflicts between them are less than one. Consider the time consuming for task 1 and 4 is 0.4hrs, for tasks 3 and 6 is 0.3hrs, for tasks 2 and 5 is 1hrs, for tasks 7 and 8 is 0.2hrs. Also, Task \((2, 5), (5, 6), (6, 7)\) conflict together with 0.1 hrs.
Task, \(((1, 2), (1, 4), (1, 5), (2, 4), (4, 8))\) conflict together with 0.4 hrs.
Task \(((1, 3), (2, 8), (3, 4), (4, 5), (5, 7))\) conflict together with 0.3 hrs.

Now, we define the fuzzy graph for above problem.

Let \(\hat{G} = (V, \sigma, \mu)\) where is the set of all task, \(\sigma(x)\) is the amount of consuming time of machine for each \(x \in V\) and \(\mu(x, y)\) is the measure of the conflict between the task \(x\) and \(y\). Finding the minimum value of job completion time for this problem is equivalent to the chromatic sum of \(\hat{G}\).

The fuzzy graph \(\hat{G} = (V, \sigma, \mu)\) corresponding to our example is defined as follows:

Let \(V = \{1, 2, 3, 4, 5, 6, 7, 8\}\),

\[
\sigma(i) = \begin{cases} 
0.4 & \text{for } i = 1, 4 \\
1.0 & \text{for } i = 2, 5 \\
0.3 & \text{for } i = 3, 6 \\
0.2 & \text{for } i = 7, 8 
\end{cases}
\]

\[
\mu(i, j) = \begin{cases} 
0.1 & \text{for } i, j \in \{(2,5), (5,6), (6,7)\} \\
0.4 & \text{for } i, j \in \{(1,2), (1,4), (1,5), (2,4), (4,7), (4,8)\} \\
0.3 & \text{for } i, j \in \{(1,3), (2,8), (3,4), (4,5), (5,7)\} 
\end{cases}
\]

The fuzzy graph for above problem is

![Fuzzy Graph Diagram](image)

From the table below, we can see that \(\Gamma_1\) satisfies all the properties of k-fuzzy coloring.

Therefore \(G\) has 8-Coloring and \(\chi^f (G) = 8\). For this 8-Coloring, chromatic number can be calculated as follows:

\[
\begin{align*}
\gamma_1 &= \{2\}, \gamma_2 = \{5\}, \gamma_3 = \{1\}, \gamma_4 = \{4\}, \\
\gamma_5 = \{3\}, \gamma_6 = \{6\}, \gamma_7 = \{7\}, \gamma_8 = \{8\} \\
\theta_1(2) &= \max\{1+0\} = 1, \theta_5(5) = \max\{1+0\} = 1, \\
\theta_4(4) &= \max\{0.4+0\} = 0.4, \theta_7(7) = \max\{0.2+0\} = 0.2 \\
\theta_3(3) &= \max\{0.3+0\} = 0.3, \theta_6(6) = \max\{0.3+0\} = 0.3, \\
\theta_8(8) &= \max\{0.2+0\} = 0.2
\end{align*}
\]

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The chromatic fuzzy sum of $G$

$$\Sigma_{\chi}(G) = 1(1) + 2(1) + 3(0.4) + 4(0.4) + 5(0.3) + 6(0.3) + 7(0.2) + 8(0.2)$$

$$= 12.1$$

Let $\Gamma_2 = \{\gamma_1, \gamma_2, \ldots, \gamma_5\}$ be a family of fuzzy set defined on $V$ where

$$\gamma_1(i) = \begin{cases} 
1.0, & \text{for } i = 5 \\
0.3, & \text{for } i = 3 \\
o, & \text{otherwise}
\end{cases}$$

$$\gamma_2(i) = \begin{cases} 
1.0, & \text{for } i = 2 \\
o, & \text{otherwise}
\end{cases}$$

$$\gamma_3(i) = \begin{cases} 
0.4, & \text{for } i = 1 \\
0.2, & \text{for } i = 7,8 \\
o, & \text{otherwise}
\end{cases}$$

$$\gamma_4(i) = \begin{cases} 
0.4, & \text{for } i = 4 \\
o, & \text{otherwise}
\end{cases}$$

$$\gamma_5(i) = \begin{cases} 
0.3, & \text{for } i = 6 \\
o, & \text{otherwise}
\end{cases}$$

Again from the table below, we can see that $\Gamma_2$ satisfied all the properties of k-fuzzy coloring.

Therefore $G$ has 5-Coloring and $\chi^f(G) = 5$. For this 5-Coloring, chromatic number can be calculated as follows:

$C_1 =\{2, 3, 5, 6\}, C_2 =\{1, 7, 8\}, C_3 =\{4\}, C_4 =\{\}$

$\theta_1(3) = \max(0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3) = 0.3$, 

$\theta_2(5) = \max(1 + 0.1 + 1 + 0.1 + 0.1 + 0.1 + 1 + 0.1) = 1.1$, 

$\theta_3(2) = \max(1 + 0.1 + 1 + 0.1 + 0.1 + 0.1 + 1 + 0.1) = 1.1$, 

$\theta_4(6) = \max(0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4) = 0.5$, 

$\theta_5(7) = \max(0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2) = 0.2$, 

$\theta_6(8) = \max(0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2) = 0.2$, 

$\theta_7(4) = \max(0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4) = 0.4$, 

$\theta_8(6) = \max(0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3) = 0.3$

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Let $\Gamma_3 = \{\gamma_1, \gamma_2, \gamma_3\}$ be a family of fuzzy set defined on $V$ where

$$\gamma_1(i) = \begin{cases} 
1.0, & \text{for } i = 2.5 \\
0.3, & \text{for } i = 3.6 \\
o, & \text{otherwise}
\end{cases}$$

$$\gamma_2(i) = \begin{cases} 
0.4, & \text{for } i = 1 \\
0.2, & \text{for } i = 7,8 \\
o, & \text{otherwise}
\end{cases}$$

$$\gamma_3(i) = \begin{cases} 
0.4, & \text{for } i = 4 \\
o, & \text{otherwise}
\end{cases}$$

Again from the table below, we can see that $\Gamma_3$ satisfied all the properties of k-fuzzy coloring.

Therefore $G$ has 3-Coloring and $\chi^f(G) = 5$. For this 3-Coloring, $\Gamma_3$ chromatic number can be calculated as follows:

$C_1 =\{3, 5, 6\}, C_2 =\{1, 7, 8\}, C_3 =\{4\}, C_4 =\{\}$

$\theta_1(3) = \max(0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3) = 0.3$, 

$\theta_2(5) = \max(1 + 0.1 + 1 + 0.1 + 0.1 + 1 + 0.1 + 1) = 1.1$, 

$\theta_3(2) = \max(1 + 0.1 + 1 + 0.1 + 0.1 + 1 + 0.1 + 1) = 1.1$, 

$\theta_4(6) = \max(0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4) = 0.5$, 

$\theta_5(7) = \max(0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2) = 0.2$, 

$\theta_6(8) = \max(0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2) = 0.2$, 

$\theta_7(4) = \max(0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4 + 0.4) = 0.4$, 

$\theta_8(6) = \max(0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3) = 0.3$

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<th>$V$</th>
<th>$\gamma_1$</th>
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</table>
The $\Gamma_3$ chromatic fuzzy sum of $G$

$$\Sigma_{\Gamma_3}(G) = 1(1.1 + 1.1 + 0.5 + 0.3) + 2(0.2 + 0.2 + 0.4) + 3(0.3)$$

$$= 5.5$$

Therefore the fuzzy chromatic sum of $G$ is

$$\Sigma(G) = \min\{\Gamma_1, \Gamma_2, \Gamma_3\}$$

$$= \min\{12.1, 8.4, 5.5\}$$

$$= 5.5$$

Calculation for $w$:

$$w = \min\{0.4 + 0, 0.4 + 0, 0.4 + 0, 0.1 + 0, 0.1 + 0, 0.1 + 0\}$$

$$= 0.2$$

Lower bound of $\Sigma(G)$ is

$$w\sqrt{8}e = 0.2\sqrt{8}\times 12 = 0.2 \times 4 \times \sqrt{6} = 1.9592$$

Now,

$$3/4[(\chi'(G)+1)h(\sigma)|V| = 3/4 [(4+1)\times 1\times 8] = 30$$

**CONCLUSION**

The fuzzy chromatic number lies between 30 and 1.9592. In our problem $\Sigma(G) = 5.5$

Therefore the minimum time of task completion of our problem is 5.5 hrs.

**REFERENCES**